Positive Coalgebraic Logic

Fredrik Dahlqvist[1]*and Alexander Kurz[2]

[1] University College London
f.dahlqvist@ucl.ac.uk
[2] University of Leicester
ak155@le.ac.uk

Partially ordered structures are ubiquitous in theoretical computer science: knowledge representation, abstract interpretation in static analysis, resource modelling, protocol or access rights modelling in formal security, etc, the list of applications is enormous. Being able to formally reason about transition systems over posets therefore seems important. The natural formalism to reason about transition systems is undoubtedly the class of *modal* logics, but most are tailored to transition structures over *sets*. This is a direct consequence of the fact that most modal logics are *boolean*. *Positive modal logic* is the exception, and is most naturally interpreted in partially ordered Kripke structures (see for example [2, 6]).

Arguably, the most natural and powerful framework to study boolean modal logics in uniform and systematic way, is the theory of *Boolean Coalgebraic Logics* (henceforth BCL, see e.g. [3]). In its abstract flavour, it is parametrised by an endofunctor $L : \mathbf{BA} \to \mathbf{BA}$ which builds modal terms over a boolean structure, an endofunctor $T : \mathbf{Set} \to \mathbf{Set}$ which builds the transition structures over which the modal terms are to be interpreted, and a natural transformation $\delta : L\mathcal{P} \to \mathcal{P}T$ (where $\mathcal{P} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{BA}$ is the powerset functor) which implements the interpretation by associating sets of acceptable successors states to each modal term over a predicate. This data, and the dual adjunction between $\mathbf{Set}$ and $\mathbf{BA}$, is traditionally summarized in the following diagram

$$\begin{tikzcd} \mathbf{BA} \arrow[loop left, "L"] \arrow[rr, bend left, "\mathcal{S}"] \arrow[rr, phantom, "\perp"] & & \mathbf{Set}^{\mathrm{op}} \arrow[loop right, "T^{\mathrm{op}}"] \arrow[ll, bend left, "\mathcal{P}"] \end{tikzcd} \tag{1}$$

where $\mathcal{S}$ is the functor sending a boolean algebra to the set of its ultrafilters.

To develop an equally powerful framework for reasoning about transition structures over posets, it seems natural to study *Positive Coalgebraic Logics* (henceforth PCL). In fact, work in this direction has already started, see for example [7, 1]. We pursue this work further and present PCL in full generality, i.e. at the same level of generality as its boolean counterpart. Moreover, given the close kinship between the two theories, we will show that the wheel does not have to be re-invented every time, and that many BCLs have a canonical positive fragment which inherits useful properties of its boolean parent. In fact, adapting well-known situations from the boolean to the positive setting is one of the guiding principles of this work.

Let us sketch the main features of the theory of PCLs and its relationship with BCLs. First of all, whilst the mathematical universe hosting the theory of BCLs is ordinary category theory, the most natural environment to discuss PCLs is category theory *enriched over* $\mathbf{Pos}$, the category of posets and monotone maps. Indeed, on the model side the category $\mathbf{Pos}$ is naturally enriched over itself, while on the syntax side we will consider endofunctors $L' : \mathbf{DL} \to \mathbf{DL}$ over the category of distributive lattice, which is also $\mathbf{Pos}$-enriched. Moreover, whilst boolean modal

*Supported by the ERC project *ProFoundNet - Probabilistic Foundations for Networks* (grant number 679127).

logics are axiomatized by equations, represented categorically as *coequalizers*, the axioms of standard positive modal logic (see [5, 2, 6]) are given by inequations, which are very naturally interpreted as *coinserters*, the **Pos**-enriched analogue of coequalizers. In this **Pos**-enriched framework, we have an analogue of Diagram (1) given by

$$\begin{tikzcd} \mathbf{DL} \arrow[loop left, "L'"] \arrow[r, bend left, "\mathcal{S}'"] \arrow[r, phantom, "\perp"] & \mathbf{Pos}^{\mathrm{op}} \arrow[l, bend left, "\mathcal{P}'"] \arrow[loop right, "(T')^{\mathrm{op}}"] \end{tikzcd}$$

where $\mathcal{S}'$ is the functor sending a distributive lattice to the poset of its prime filters (under inclusion), and $\mathcal{P}'$ is the functor sending a poset to the distributive lattice of its upsets.

Coalgebras over posets. Transition structures over posets will be formalized as coalgebras for an endofunctor $T' : \mathbf{Pos} \to \mathbf{Pos}$. Here, we are already confronted with a situation which perfectly captures the philosophy of this work. How do we choose such a functor? In practice both our requirements and our intuition are guided by examples of endofunctors $T : \mathbf{Set} \to \mathbf{Set}$, for example non-deterministic computations modelled as coalgebras for the powerset functor $\mathsf{P} : \mathbf{Set} \to \mathbf{Set}$, or models of graded modal logic as coalgebras for the multiset functor $\mathsf{B} : \mathbf{Set} \to \mathbf{Set}$. The solution is to adapt these well-know functors to posets. We use the *posetification* procedure developed in [1] and define for each **Set**-endofunctor T its posetification by $T' : \mathbf{Pos} \to \mathbf{Pos}$ by $T' = \mathrm{Lan}_{\mathsf{D}}\mathsf{D}T$, where D is the functor sending a set to its discrete poset. It was shown in [1] that T' can be computed using certain coinserters, and we now have a whole repertoire of **Set**-endofunctors for which we have computed the posetification: the powerset functor, the neighbourhood functor, the monotone neighbourhood functor, the multiset functor, etc, as well as a grammar to combine them.

Syntax. The syntax of a positive coalgebraic logic will be given by a locally monotone (i.e. **Pos**-enriched) endofunctor $L' : \mathbf{DL} \to \mathbf{DL}$. Whilst [1] focused defining such functors directly from the semantics, here we once again focus on adapting existing boolean logics. This leads us to an operation which is dual to that of posetification, and which we call *positivisation*: given an endofunctor $L : \mathbf{BA} \to \mathbf{BA}$, we define its positivisation $L' : \mathbf{DL} \to \mathbf{DL}$ by $L' = \mathrm{Ran}_{\mathsf{U}}\mathsf{U}L$ where $\mathsf{U} : \mathbf{BA} \to \mathbf{DL}$ is the obvious forgetful functor. The positivisation of a **BA**-endofunctor can be computed explicitly via *inserters*. We can distinguish two classes of positivisation: those for which the natural transformation $\beta : L'\mathsf{U} \to \mathsf{U}L$ given by universality of the right (enriched) Kan extension is an iso, and all the others. The former correspond to boolean logics $L : \mathbf{BA} \to \mathbf{BA}$ which have a *monotone* presentation. We have computed the positivisation of the functors defining the boolean modal logics with (i) no axioms, (ii) monotonicity only, (iii) the standard axioms of modal logic, and (iv) the axioms of graded modal logic.

Semantics. Following our guiding philosophy, we would like to canonically turn a boolean coalgebraic logic (L, T, δ) with nice properties (for example completeness) into a positive coalgebraic logic (L', T', δ'), hopefully with equally nice properties. First we need to build a semantic natural transformation $\delta' : L'\mathcal{P}' \to \mathcal{P}'T'$, from the transformation $\delta : L\mathcal{P} \to \mathcal{P}T$. By combining the posetification and the positivisation procedures described above, and the properties of $\mathcal{P}'$, one can build a transformation $\delta' : L'\mathcal{P}' \to \mathcal{P}'T'$ from δ in a universal way. Moreover, if δ and $\beta : L'\mathsf{U} \to \mathsf{U}L$ are component-wise injective, so is δ', in other words completeness transfers from the boolean logic to its canonical positive fragment. Similarly, strong completeness via the coalgebraic Jónsson-Tarski theorem – which is equivalent to the adjoint (or mate) $\hat{\delta}$ of δ being component-wise split epi [8, 9, 4] – transfers from a boolean coalgebraic logic to its positive fragment. The transfer of expressivity is more involved.

References

[1] A. Balan, A. Kurz, and J. Velebil. Positive fragments of coalgebraic logics. In *International Conference on Algebra and Coalgebra in Computer Science*, pages 51–65. Springer, 2013.

[2] S. Celani and R. Jansana. A new semantics for positive modal logic. *Notre Dame Journal of Formal Logic*, 38(1), 1997.

[3] C. Cirstea, A. Kurz, D. Pattinson, L. Schröder, and Y. Venema. Modal logics are coalgebraic. *The Computer Journal*, 54(1):31–41, 2009.

[4] F. Dahlqvist. Coalgebraic completeness-via-canonicity - principles and applications. In I. Hasuo, editor, *CMCS 2016*, volume 9608 of *LNCS*, pages 174–194. Springer, 2016.

[5] J. M. Dunn. Positive modal logic. *Studia Logica*, 55(2):301–317, 1995.

[6] M. Gehrke, H. Nagahashi, and Y. Venema. A Sahlqvist theorem for distributive modal logic. *Annals of pure and applied logic*, 131(1):65–102, 2005.

[7] K. Kapulkin, A. Kurz, and J. Velebil. Expressiveness of positive coalgebraic logic. *Advances in Modal Logic*, 9:368–385, 2012.

[8] C. Kupke, A. Kurz, and D. Pattinson. Ultrafilter Extensions for Coalgebras. In J. L. Fiadeiro, N. Harman, M. Roggenbach, and J. J. M. M. Rutten, editors, *Algebra and Coalgebra in Computer Science (CALCO 2005)*, volume 3629, pages 263–277. Springer, 2005.

[9] A. Kurz and J. Rosický. Strongly complete logics for coalgebras. *Logical Methods in Computer Science*, 8, 2012.